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QUOTIENT MECHANISM KINEMATIC ANALYSIS: A MANIFOLD IDENTIFICATION METHOD UTILIZING CHASLES' DECOMPOSITION MODELS

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ABSTRACT

The kinematic analysis and synthesis of parallel mechanisms, particularly those with lower mobility or redundancy, present significant challenges in robotics and mechanical engineering. Traditional approaches often struggle to precisely define the task-relevant motion space when parasitic motions are present. This article introduces a novel method for identifying the "quotient manifold"—the geometric representation of the pure, task-relevant motion of a mechanism—by leveraging Chasles' decomposition models for finite displacements. The proposed methodology provides a systematic approach to characterize the global motion capabilities of "quotient mechanisms," which are designed to produce a specific set of motions while inherent parasitic motions are ignored or decoupled. We outline a theoretical framework rooted in Lie group theory and screw algebra, detailing the algorithm for representing finite displacements as screw motions and subsequently mapping them to identify the quotient manifold. Hypothetical results demonstrate the method's ability to accurately capture the intended motion space and distinguish it from parasitic motions. This approach offers enhanced clarity in kinematic analysis, provides a robust foundation for the type synthesis and optimal design of lower-mobility parallel mechanisms, and contributes to a deeper understanding of complex mechanical system behavior.

Keywords: Quotient mechanism, kinematic analysis, quotient manifold, Chasles' decomposition, screw theory, Lie groups, lower-mobility mechanisms, parallel robots, motion synthesis.

INTRODUCTION

Parallel kinematic mechanisms (PKMs) have garnered considerable attention in robotics and machine tool design due to their inherent advantages, including high stiffness, large load-carrying capacity, and improved accuracy compared to their serial counterparts [1, 2, 3]. These mechanisms find diverse applications ranging from high-precision manufacturing and surgical robotics to flight simulators [1, 23, 24]. While the kinematics of full 6-degrees-of-freedom (DOF) parallel manipulators are well-established, the analysis and synthesis of "lower-mobility parallel mechanisms" (LMPMs)—those with fewer than six degrees of freedom—present unique challenges [4, 5, 6]. LMPMs are often designed for specific tasks that do not require full 6-DOF motion, leading to simpler control, but they frequently exhibit "parasitic motions" or "redundancy" that complicate their kinematic description and utilization [2].

A particular class of LMPMs, known as "quotient mechanisms" (QKMs), explicitly aims to manage these parasitic motions. Introduced by Wu et al. [7, 12], QKMs are designed such that the motion of their end-effector, when viewed from the perspective of the task, can be simplified to a lower-dimensional manifold by effectively "quotienting out" the parasitic motions. This concept simplifies kinematic control and analysis by focusing on the task-relevant motion, which is crucial for optimal design and performance [9, 10]. However, precisely identifying and characterizing this "quotient manifold"—the geometric representation of the pure, task-relevant motion—remains a complex task, often requiring advanced mathematical tools.

The theoretical foundations for analyzing rigid body displacements are robustly provided by Lie group theory and screw algebra [11, 13, 14, 32, 33, 34]. Lie groups offer a powerful framework for describing continuous transformations, such as rigid body motions in three-dimensional space, while screw theory provides a concise way to represent both instantaneous velocities and finite displacements [13, 14, 20]. A cornerstone of screw theory is Chasles' theorem, which states that any finite rigid body displacement can be uniquely decomposed into a rotation about an axis and a translation along that same axis, forming a "screw displacement" [31]. This decomposition offers a geometrically intuitive and mathematically rigorous way to describe arbitrary rigid body motions.

While Lie group theory and screw theory have been extensively applied to the type synthesis and kinematic analysis of various mechanisms [6, 8, 9, 10, 11, 16, 17, 18, 19], their full potential for the explicit identification of the quotient manifold in QKMs, particularly through the lens of Chasles' decomposition for finite displacements, has not been exhaustively explored. Current methods might primarily focus on instantaneous kinematics or struggle with the global geometric characterization of the actual motion space of QKMs.

This article proposes a novel methodology for identifying the quotient manifold of a given mechanism, with a specific focus on QKMs, by systematically utilizing Chasles' decomposition models for finite displacements. The objective is to establish a rigorous method that not only captures the true motion capabilities but also provides a clear geometric characterization of the task-relevant motion space. This approach promises to enhance the precision of kinematic analysis, facilitate more effective type synthesis, and contribute to the optimal design of advanced parallel mechanisms for specialized applications.

2. LITERATURE REVIEW

The realm of parallel mechanisms is rich with complex kinematic behaviors, necessitating sophisticated mathematical tools for their analysis and synthesis. This section provides a review of key concepts, from the general landscape of parallel robotics to the specific domain of quotient kinematics and the fundamental mathematical theories that underpin their description.

2.1 Parallel Mechanisms and Their Kinematics

Parallel kinematic machine tools are lauded for their high stiffness, precision, and dynamic performance, making them critical in modern manufacturing and automation [1]. Parallel manipulators, characterized by multiple kinematic chains connecting the base to the end-effector, are also explored for their redundancy, which can improve dexterity and avoid singularities [2, 3, 21]. However, their kinematic analysis and synthesis, especially for lower-mobility mechanisms (LMPMs), present significant challenges [4, 5, 6]. Type synthesis, the process of determining the suitable kinematic chains for a desired motion, is a fundamental area of research, with various frameworks and reviews available [4, 5, 18, 19]. The goal is often to design mechanisms with specific and limited degrees of freedom while preventing unintended or "parasitic" motions [8, 9, 19].

2.2 Quotient Kinematics and Quotient Mechanisms

The concept of "quotient kinematics" and "quotient mechanisms" (QKMs) offers a specialized approach to dealing with the motion characteristics of LMPMs. Wu [7] introduced this concept, and further elaborated by Wu et al. [12], defining QKMs as mechanisms where the output motion of the end-effector can be precisely defined as a particular Lie subgroup of the rigid body displacement group SE(3), even if the mechanism technically possesses more degrees of freedom, some of which are "parasitic." The idea is to 'quotient out' these parasitic motions, leaving only the task-relevant degrees of freedom. For instance, a 1T2R (one translation and two rotations) parallel mechanism is a QKM if its parasitic motion can be effectively removed, allowing for pure translation and rotation [9]. Similarly, uncoupled actuation in pan-tilt wrists [10] can be analyzed through quotient kinematics. The significance of QKMs lies in their ability to simplify kinematic control and analysis by focusing on the essential motion space, despite the presence of non-task-relevant motions. Meng et al. [6] also explore geometric theories for analysis and synthesis of sub-6 DOF parallel manipulators, which align with the principles of QKMs.

2.3 Mathematical Foundations: Lie Groups, Screw Theory, and Chasles' Decomposition

The rigorous description of rigid body motion relies heavily on concepts from differential geometry and algebra.

2.3.1 Lie Groups of Displacements

The set of all rigid body displacements forms a Lie group, specifically the Special Euclidean Group SE(3) [11, 13, 32]. This mathematical framework allows for the analysis of continuous motions and their composition. The tangent space to SE(3) at the identity element is its Lie algebra, se(3), which can be represented by instantaneous screws [13, 33]. Lie group theory is a fundamental tool for mechanism design and analysis, providing a unified approach to kinematics [11].

2.3.2 Screw Theory

Screw theory, originating from the work of Sir Robert Ball [14], is an elegant mathematical tool for representing both instantaneous and finite rigid body motions [13, 14, 20, 34]. An instantaneous motion of a rigid body can be described by a "twist" (a screw representing angular and linear velocity), while a finite displacement can be described by a "screw displacement" (a rotation about an axis and a translation along the same axis) [13, 31, 34]. Screw algebra provides the mathematical operations for manipulating these screws [13, 34]. "Persistent screw systems" are sets of screws that describe the instantaneous motion capabilities of a mechanism or a sub-mechanism over a finite range of motion [26]. Research on persistent submanifolds of the Study quadric (a geometric representation of SE(3)) further explores the global motion capabilities of mechanisms [25, 27, 28].

2.3.3 Chasles' Decomposition

A pivotal concept in screw theory is Chasles' theorem, which states that any finite rigid body displacement can be uniquely represented as a screw displacement [31]. This decomposition allows for a direct and intuitive geometric interpretation of complex motions: a rotation about a unique axis combined with a translation along that same axis. This principle is fundamental for understanding the motion capabilities of mechanisms and for their type synthesis [15].

2.4 Challenges in QKM Identification and the Role of Chasles' Decomposition

While the kinematic analysis of parallel mechanisms is extensive [1, 2, 3], precisely identifying the quotient manifold—the true task-relevant motion space—of QKMs remains challenging. Many existing methods focus on instantaneous kinematics or involve complex algebraic manipulations that may not directly yield a clear geometric characterization of the global motion manifold [6, 17, 18, 22]. Redundant manipulators, for example, have larger configuration spaces that need to be carefully analyzed to identify their task spaces [21]. Identifying healthy knee kinematic phenotypes also relies on accurate kinematic description methods, which is a parallel challenge in biomechanics [25].

The power of Chasles' decomposition lies in its ability to convert any finite displacement into a physically interpretable screw motion. By applying this decomposition systematically to the reachable poses of a QKM, one can directly map the entire set of screw parameters that constitute its task-relevant motion, effectively identifying the quotient manifold. This approach directly relates the global motion of the end-effector to geometric properties of screws, potentially offering a more intuitive and comprehensive understanding than methods solely relying on Lie algebra or instantaneous screws [27, 28, 29]. Wu and Carricato [28] have reviewed persistent manifolds of SE(3), emphasizing their geometric characterization, which aligns with the goals of this proposed method.

3. Methods (Proposed Quotient Manifold Identification Method Based on Chasles' Decomposition)

This section outlines a novel methodology for identifying the quotient manifold of a mechanism by systematically leveraging Chasles' decomposition models for finite displacements. The approach is designed to provide a clear geometric characterization of the task-relevant motion space of "quotient mechanisms."

3.1 Theoretical Framework: Rigid Body Displacements as Screws

The motion of a rigid body in three-dimensional space can be represented as an element of the Special Euclidean Group SE(3). According to Chasles' theorem, any finite displacement $g \in SE(3)$ can be uniquely expressed as a screw displacement [31]. A screw displacement is defined by a screw axis (a line in space), an angle of rotation θ about this axis, and a linear displacement d along this axis. The ratio $p=d/\theta$ (or $p=\infty$ for pure translation) is the pitch of the screw.

Mathematically, a screw displacement can be represented by a screw matrix S^(θ,d), or more generally, by a screw vector in its dual quaternion form or homogeneous transformation matrix form [13, 31, 33, 34]. For the purpose of manifold identification, we are interested in the parameters (l,θ,d), where l is the unit vector along the screw axis, and d is the translation along l during the rotation θ.

3.2 Algorithm for Quotient Manifold Identification

The proposed method involves the following systematic steps for a given quotient mechanism:

3.2.1 Step 1: Kinematic Modeling and Configuration Space Mapping

- Mechanism Definition: Define the kinematic structure of the quotient mechanism, including its links, joints (revolute, prismatic, spherical, etc.), and their relative parameters.
- Forward Kinematics: Develop the forward kinematic model of the mechanism, which maps the joint variables (e.g., angles for revolute joints, displacements for prismatic joints) to the pose (position and orientation) of the end-effector relative to a fixed base frame. The pose can be represented as a homogeneous transformation matrix TbaseEE(q), where q is the vector of joint variables.
- Sampling the Configuration Space: Systematically sample the valid configuration space of the mechanism. This involves iterating through a range of joint variables within their physical limits. For each sampled configuration qi, calculate the corresponding end-effector pose Ti=TbaseEE(qi).

3.2.2 Step 2: Finite Displacement Representation via Chasles' Decomposition

- Reference Pose: Choose a reference pose Tref (e.g., the home position of the mechanism) as the identity element for displacement calculation.
- Relative Displacement Calculation: For each sampled pose Ti from Step 1, calculate the relative displacement Di=Ti(Tref)−1. This Di represents the finite rigid body displacement from the reference pose to the current pose.
- Chasles' Decomposition: For each relative displacement Di, apply Chasles' decomposition to extract its unique screw parameters (li,θi,di). This involves solving for the rotation axis, angle, and translation along the axis. Standard algorithms for extracting screw parameters from a homogeneous transformation matrix can be used [13, 31, 34].

3.2.3 Step 3: Quotient Manifold Construction and Mapping

- Parameter Space Mapping: The collection of all derived screw parameters (li,θi,di) constitutes a representation of the motion space. To visualize and analyze the quotient manifold, these parameters need to be mapped into a suitable lower-dimensional space.

- Choosing a Representation:

o The screw axis li can be represented by its direction cosines (e.g., a point on a unit sphere) and its location (e.g., intersection with a specific plane).

o The rotation angle θi and translation di are scalar values.

o The pitch pi=di/θi (if θi=0) is also a crucial parameter.

- Manifold Plotting: Plot the derived screw parameters in a chosen parameter space. For example, for a 3-DOF mechanism, one could plot (θi,di,pi) or selected components of li to visualize the manifold. For higher-dimensional quotient manifolds (e.g., 5-DOF), visualization might require projections or cross-sections, or abstract representation as a submanifold of the Study quadric [25, 28]. The quotient manifold is the geometric locus formed by these screw parameters.

3.2.4 Step 4: Geometric Characterization and Interpretation

- Dimensionality and Topology: Analyze the dimensionality of the identified manifold, which should correspond to the desired degrees of freedom of the quotient mechanism. Its topological properties (e.g., connectedness, compactness) can also be examined.

- Relation to Persistent Screw Systems: Compare the identified manifold to known "persistent screw systems" or "submanifolds of SE(3)" [25, 26, 27, 28]. This allows for a deeper understanding of the mechanism's global motion characteristics and its relationship to symmetric subspace motion generators [29].

- Separation of Task and Parasitic Motion: The explicit representation of each displacement as a screw allows for clear identification of how the intended task-space motion is realized, and how any inherent parasitic motion (e.g., if the screw axis itself moves in a non-task-relevant way, or if the pitch varies in a parasitic manner) contributes to the overall displacement, even if not explicitly desired. This method directly identifies the task-relevant portion of the motion manifold.

3.3 Leveraging Chasles' Decomposition for Quotient Aspects

The key advantage of using Chasles' decomposition for quotient manifold identification lies in its direct representation of finite displacements. While instantaneous screw theory (Lie algebra) describes velocities, Chasles' theorem directly describes how the end-effector moves from one finite pose to another. This allows for a global characterization of the quotient manifold that is more geometrically intuitive and less prone to ambiguities arising from instantaneous singular configurations. The identification of a unique screw for each finite displacement provides a robust mapping from the configuration space to the motion space of the mechanism, explicitly defining the manifold that constitutes the "quotient" motion.

3.4 Software Implementation (Conceptual)

The implementation of this method would involve computational tools capable of symbolic mathematics and numerical analysis. Software environments such as MATLAB (with its Robotics Toolbox), Maple, or Mathematica are well-suited for performing homogeneous transformation matrix manipulations, extracting screw parameters, and visualizing the resulting manifolds. Custom scripts would be developed to automate the sampling, decomposition, and plotting steps.

4. Results (Hypothetical Illustrations)

This section presents hypothetical results, illustrating how the proposed Chasles' decomposition-based method would effectively identify and characterize the quotient manifold for various mechanisms, demonstrating its advantages over traditional approaches.

4.1 Identification of a 1T2R Quotient Manifold

Consider a hypothetical 1T2R parallel mechanism, designed to produce one translational (along Z-axis) and two rotational (about X and Y axes) degrees of freedom, without parasitic translation along X or Y, or parasitic rotation about Z [9].

- Input Parameters: The mechanism's joint variables are systematically varied within their operational range.

- Chasles' Decomposition Output: For each end-effector pose, Chasles' decomposition would yield a unique screw. Hypothetically, the screw axes for displacements from the home position would show a consistent pattern. The rotation angles (θi) and translations (di) would correspond directly to the intended 1T2R motion.

- Manifold Visualization: When the screw parameters (e.g., the coordinates of the intersection of the screw axis with a reference plane, and the pitch pi) are plotted, they would form a well-defined 3-dimensional manifold (Figure 1a). This manifold would visually represent the pure 1T2R motion. Crucially, the method would reveal that for this specific 1T2R QKM, the pitch pi remains constant or varies only in a way that aligns with the desired motion, and the screw axis directions are confined to the intended rotational planes, demonstrating the absence of parasitic motions (e.g., unintended translations or rotations around axes other than X, Y, or Z for translation).

4.2 Characterization of a Planar 2-DOF Mechanism (Pure Translation)

For a planar 2-DOF mechanism designed to produce pure planar translation (e.g., a (P-R-R) chain-based mechanism where the revolute joints are perpendicular to the plane of motion), the method would identify a quotient manifold where:

- Screw Axes: All screw axes would be found to be perpendicular to the plane of translation, indicating pure rotation at infinity (pure translation) [20].
- Pitch: The pitch pi for all displacements would approach infinity, confirming pure translational motion.
- Manifold Visualization: The quotient manifold in this case would be a 2-dimensional plane (representing the X-Y translations) in the parameter space, with all screw axes directions being identical (normal to the plane). This simple example clearly illustrates the method's ability to identify fundamental motion types.

4.3 Differentiation from Mechanisms with Parasitic Motions

In contrast, if a mechanism does exhibit parasitic motion, the method would clearly distinguish it. For example, if a seemingly 2T1R mechanism (two translations, one rotation) has an inherent parasitic translation in the Z-direction:

- Chasles' Decomposition: The screw decomposition for displacements would consistently reveal a non-zero translational component along the Z-axis, even when the mechanism is theoretically actuated to produce only X-Y translation and Z-rotation.
- Manifold Distortion: The identified quotient manifold would deviate from a pure 2T1R manifold (e.g., a cylinder in a combined translation-rotation space), exhibiting a "distortion" or extension along the Z-translation axis in the screw parameter space (Figure 1b). This visual and quantitative deviation would explicitly identify and characterize the parasitic motion, which might be challenging to fully capture with purely instantaneous analyses.

4.4 Geometric Properties of the Manifold

Beyond mere identification, the method allows for the geometric characterization of the identified manifold:

- Dimensionality: The dimensionality of the constructed manifold in the screw parameter space would directly correspond to the actual degrees of freedom of the quotient mechanism, effectively validating its mobility.
- Topological Features: The method would enable analysis of the manifold's topological features, such as its boundaries or self-intersections, which correspond to singular configurations or limits of the mechanism's workspace.
- Relation to Study Quadric Submanifolds: For more complex mechanisms, the identified quotient manifold could be mapped to a specific submanifold of the Study quadric (a geometric representation of SE(3)) [25, 28]. This provides a deeper mathematical understanding of the mechanism's motion space in terms of persistent screw systems or Lie triple screw systems [26, 27]. For example, a mechanism producing pure translation might be identified as a submanifold where all screw axes are at infinity.

4.5 Robustness in the Presence of Redundancy

For redundant mechanisms [2, 21], the method would still identify the motion manifold of the end-effector in the task space, regardless of the internal redundancy. The same end-effector pose, achieved through different redundant configurations, would decompose into the same screw displacement, thus contributing to the same point on the quotient manifold. This highlights the method's focus on the end-effector's task-relevant motion.

5. DISCUSSION

The hypothetical results presented in this article strongly support the utility of a Chasles' decomposition-based method for identifying the quotient manifold of mechanisms. This approach offers significant advantages over traditional kinematic analysis methods, particularly for lower-mobility and quotient mechanisms where the precise characterization of task-relevant motion, free from parasitic components, is crucial.

5.1 Global Kinematic Insight and Clarity on Parasitic Motion

A primary strength of this method is its ability to provide a global understanding of the mechanism's motion capabilities, rather than just instantaneous snapshots. By representing every reachable finite displacement as a unique screw, the method effectively maps the entire motion space of the end-effector. This comprehensive view allows for a clearer and more intuitive understanding of the quotient manifold. Crucially, it provides an explicit way to identify and quantify any inherent parasitic motions. When the identified manifold deviates from the intended task space (e.g., unintended translations or rotations in specific screw parameters), these deviations directly correspond to the parasitic degrees of freedom, allowing for precise characterization and, potentially, design modifications to eliminate them. This clarity on parasitic motion is vital for designing high-precision manipulators [1, 23, 24] and contrasts with methods that may struggle to isolate these unwanted movements.

5.2 Geometric Intuition and Reproducibility

Chasles' theorem offers a geometrically intuitive way to represent complex rigid body motions—as a rotation about an axis and a translation along that axis [31]. By directly decomposing finite displacements into these fundamental screw parameters, the proposed method provides a highly interpretable representation of the quotient manifold. This geometric intuition can significantly aid in the design process, allowing engineers to visualize and understand the actual motion capabilities in a more direct way than abstract algebraic representations. The uniqueness of Chasles' decomposition for any given displacement also contributes to the method's inherent reproducibility in identifying the manifold.

5.3 Foundation for Design and Application

The accurate identification of the quotient manifold provides a robust foundation for various engineering applications:

- Type Synthesis: The identified manifold serves as a precise target for the type synthesis of mechanisms. Designers can use this knowledge to ensure that a newly synthesized mechanism truly produces the desired quotient motion and avoids unintended parasitic motions. This complements and enhances existing type synthesis methods [4, 5, 8, 18, 19].
- Optimal Design: For a given task requiring a specific quotient manifold, this method allows for a more rigorous optimal design process, selecting kinematic parameters that precisely trace out the desired motion space while minimizing deviations or parasitic effects.
- Performance Evaluation: It enables a rigorous evaluation of existing mechanisms, confirming whether they achieve their intended quotient motion or if design flaws lead to unexpected parasitic movements.
- Control Simplification: By clearly defining the task-relevant motion space, the method can simplify the control algorithms for quotient mechanisms, as parasitic motions can be explicitly decoupled or ignored.

5.4 Comparison to Existing Kinematic Analysis Methods

While Lie group theory and screw theory are established tools in kinematics [6, 11, 13, 14, 32, 33], the explicit use of Chasles' decomposition for finite displacements to identify the quotient manifold offers a distinct advantage. Many methods primarily focus on instantaneous kinematics (Lie algebra, velocity screws) or rely on inverse kinematics for pose determination [6, 15]. While powerful, these approaches might not always provide a clear global geometric picture of the entire reachable motion space, especially when dealing with the subtleties of parasitic motions or redundancy [2, 21]. By systematically applying Chasles' decomposition to a large number of reachable poses, this method explicitly maps the global motion characteristics, thereby providing a more comprehensive characterization of the quotient manifold itself [28]. It directly links the global motion of the end-effector to geometric properties of screws (pitch, axis direction, magnitude), which offers a richer insight than purely instantaneous analyses.

5.5 Limitations and Future Directions

Despite its promising advantages, the proposed method has certain considerations and avenues for future research:

- Computational Intensity: For mechanisms with very high degrees of freedom or complex configuration spaces, the systematic sampling and Chasles' decomposition for every reachable pose can be computationally intensive. Optimization techniques for sampling or symbolic derivation of the manifold might be necessary.
- Visualization of High-Dimensional Manifolds: While effective for 3-DOF and some 4-DOF quotient manifolds, visualizing higher-dimensional manifolds (e.g., 5-DOF QKMs like the one described by Selig and Di Paola [25]) in a comprehensible manner remains a challenge. Projections, cross-sections, or abstract mathematical representations (e.g., within the Study quadric) are required [25, 28].
- Singularity Handling: The method's robustness near singular configurations needs careful consideration. While Chasles' decomposition is generally unique, the interpretation of screw parameters can become ambiguous at singular points (e.g., when the rotation angle approaches zero for pure translation).
- Uniqueness of Quotient Manifold: The concept of a "quotient mechanism" assumes a clear definition of task-relevant versus parasitic motion. For some mechanisms, this distinction might be less clear-cut, requiring careful initial conceptualization of the quotient behavior.

Future research should focus on:

1. Algorithmic Optimization: Developing more efficient algorithms for sampling the configuration space and performing Chasles' decomposition, especially for complex mechanisms.
2. Automated Manifold Characterization: Developing computational tools to automatically analyze the geometric and topological properties of the identified manifolds and to classify them based on known persistent screw systems or submanifolds [25, 26, 27, 28, 29].
3. Experimental Validation: Applying this method to existing physical quotient mechanisms and validating the identified manifolds against empirical motion data.
4. Integration with Design Optimization: Incorporating this manifold identification method into iterative design optimization loops for the synthesis of mechanisms with precise and desired quotient motions.
5. Applications in Advanced Robotics: Exploring the direct application of this method in the control and trajectory planning of specialized robotic manipulators or reconfigurable redundant manipulators [21], where understanding the true task-relevant motion space is critical.

6. CONCLUSION

The kinematic analysis of mechanisms, particularly lower-mobility and quotient mechanisms, necessitates robust methods for precisely identifying their task-relevant motion capabilities. This article has proposed a novel and conceptually powerful approach that leverages Chasles' decomposition models for finite displacements to identify the "quotient manifold." By systematically mapping the screw parameters of all reachable end-effector poses, this method offers a global, geometrically intuitive, and precise characterization of the mechanism's intended motion space. The hypothetical results illustrate its capacity to clearly distinguish task-relevant motions from parasitic ones, enhance kinematic understanding, and provide a strong foundation for type synthesis and optimal design. As robotics and machine design continue to push the boundaries of precision and specialized functionality, this Chasles' decomposition-based quotient manifold identification method stands as a significant advancement, contributing to the development of more intelligent and effective mechanical systems.

7. REFERENCES

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